Quantum Bilinear Optimisation

arXiv:1506.08810

Mario Berta (IQIM Caltech), Omar Fawzi (ENS Lyon),
Volkher Scholz (Ghent University)

March 7th, 2016
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Goal: understand the power of quantum assistance and quantum adversaries for operational setups

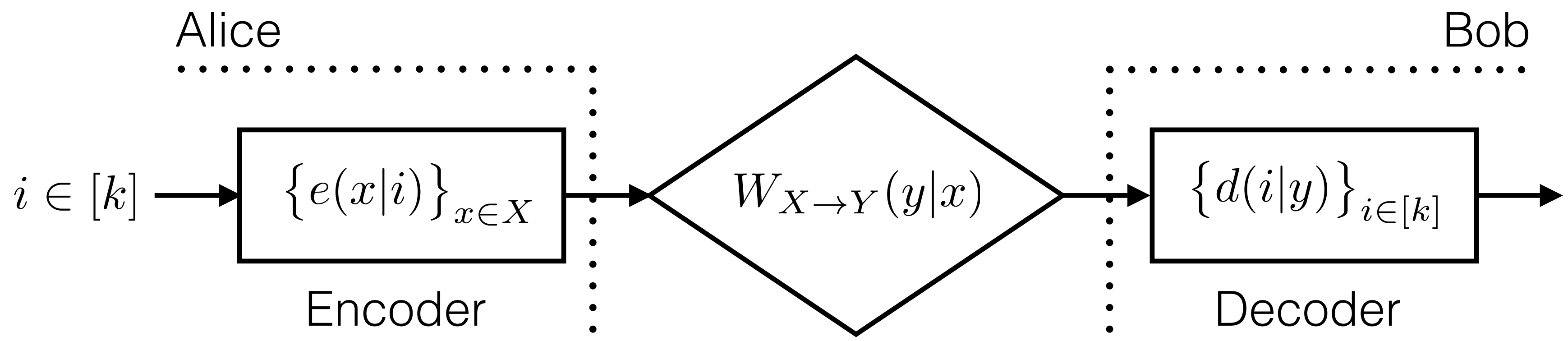
Example:

Classical noisy channel coding

(vs.)

Entanglement-assisted channel coding

Classical noisy channel coding (I)



- Given noisy channel $W_{X\to Y}$ mapping X to Y with transition probability:

$$W_{X\to Y}(y|x)\ \forall (x,y) \in X \times Y$$

- The goal is to send k different messages using W while minimising the error probability for decoding:

"bilinear optimisation"

$$\begin{aligned} p_{\text{succ}}(W,k) := \underset{(e,d)}{\text{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)e(x|i)d(i|y) \\ \text{subject to} \quad & \sum_x e(x|i) = 1 \quad \forall i \in [k]\,, \quad \sum_i d(i|y) = 1 \quad \forall y \in Y \\ & 0 \le e(x|i) \le 1 \quad \forall (x,i) \in X \times [k]\,, \quad 0 \le d(i|y) \le 1 \quad \forall (i,y) \in [k] \times Y\,. \end{aligned}$$

Classical noisy channel coding (II)

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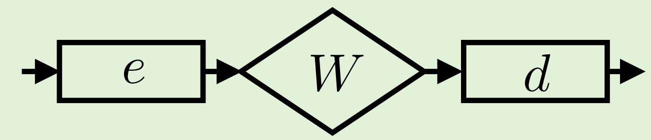
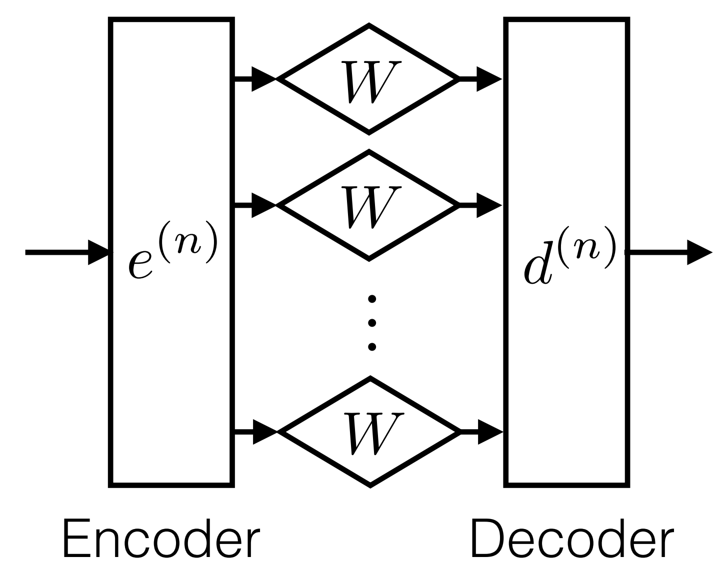


compared to

- Shannon's asymptotic independent and identical distributed (iid) channel capacity:

Definition: $C(W) := \sup\left\{R \,\middle|\, \lim_{n\to\infty} p_{\mathrm{succ}}\left(W^{\times n}, \lfloor 2^{Rn} \rfloor\right) = 1\right\}$

Answer: $C(W) = \max_{P_X} I(X:Y)$ mutual information



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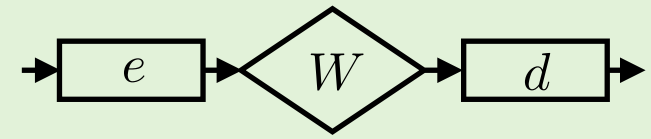

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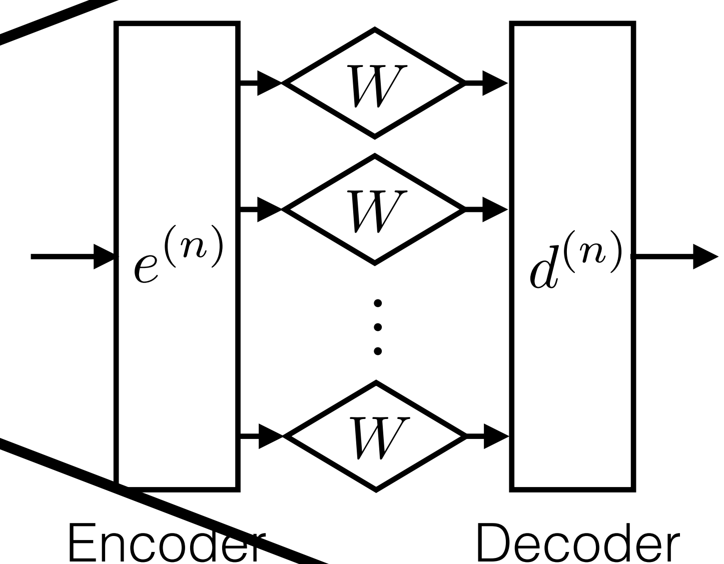

Example:

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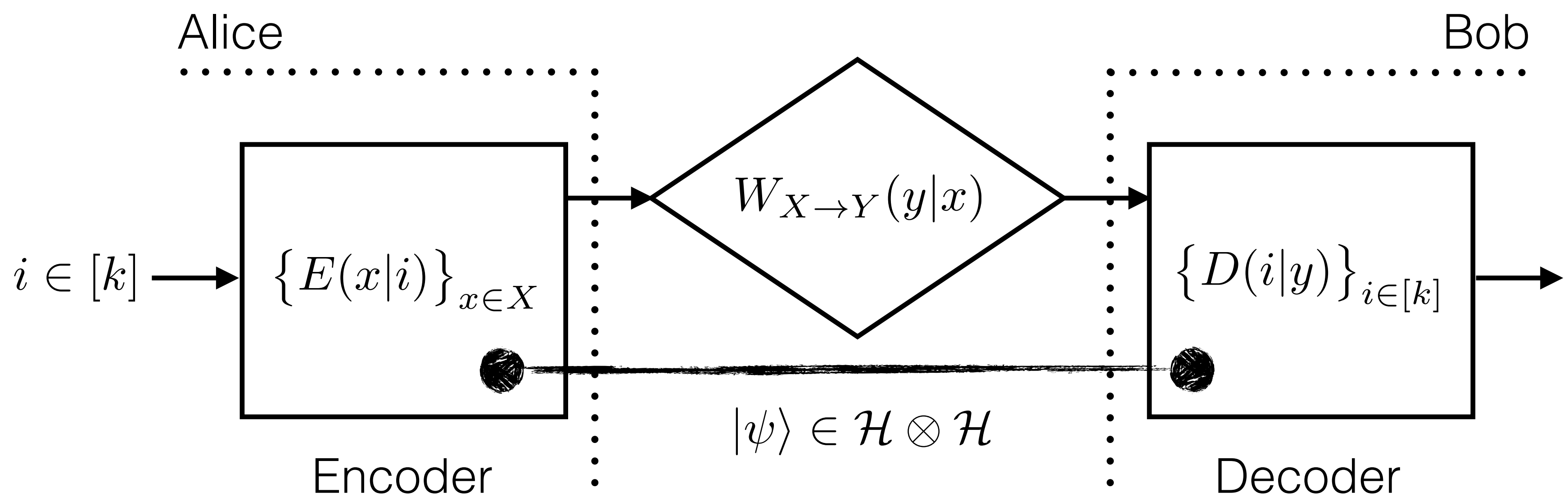
(vs.)

Entanglement-assisted channel coding

Entanglement-assisted channel coding (I)



"quantum bilinear optimisation"

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p^*_{\mathrm{succ}}(W,k) := \underset{(\mathcal{H},\psi,E,D)}{\text{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\langle\psi|E(x|i)\otimes D(i|y)|\psi\rangle \\
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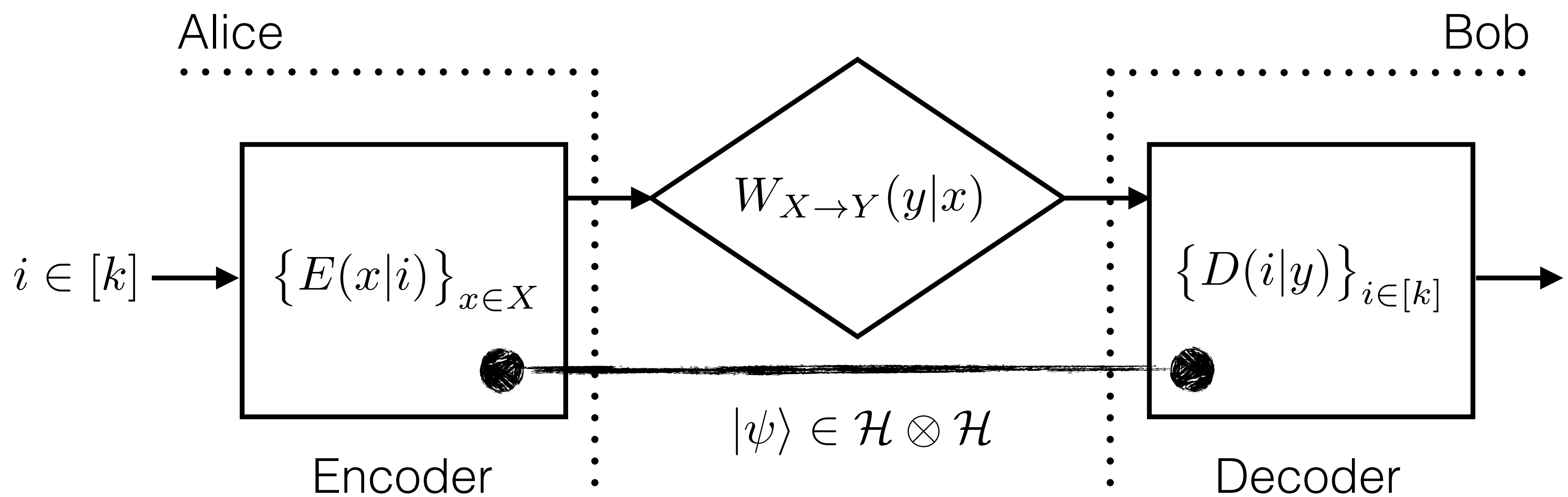

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- **Scalar** (commutative) versus **matrix** (non-commutative) variables:

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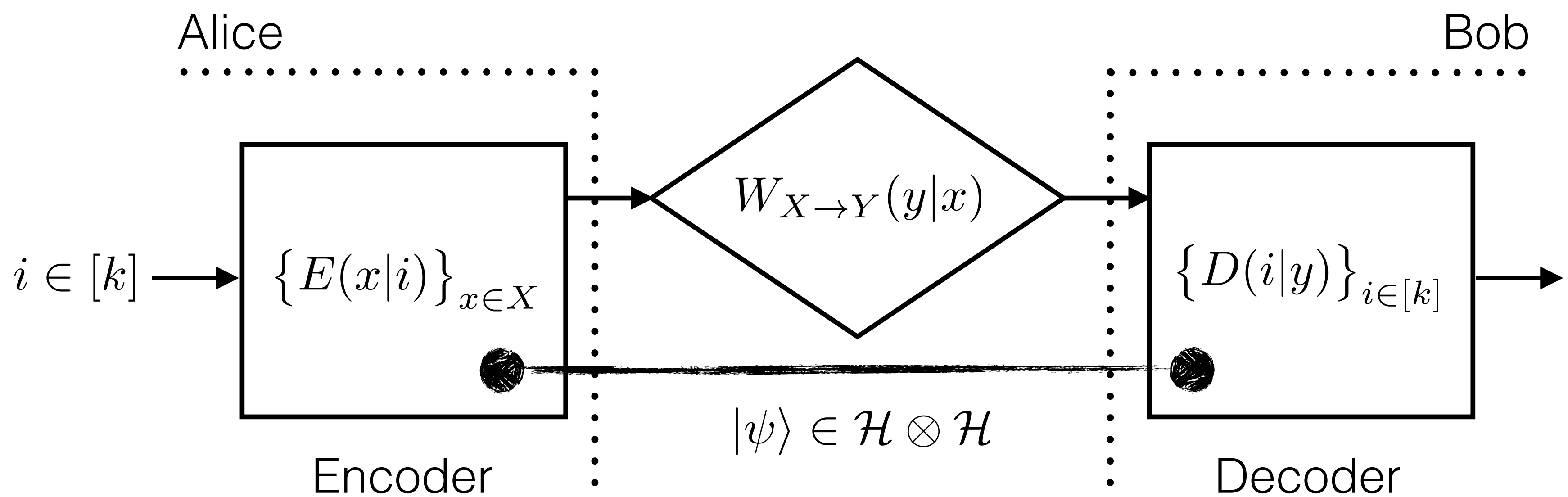

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- Unknown if $p^*_{\mathrm{succ}}(W,k)$ is computable!

Entanglement-assisted channel coding (II)

- Understand the possible separation: $p_{\mathrm{succ}}(W,k) \quad \text{versus} \quad p^*_{\mathrm{succ}}(W,k)$

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$$p_{\mathrm{succ}}(Z,2) = \frac{5}{6} \approx 0.833 \quad \text{vs.} \quad p^*_{\mathrm{succ}}(Z,2) \geq \frac{2+2^{-1/2}}{3} \approx 0.902$$

[Prevedel et al., PRL (2011)]

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—> this is also optimal with two-dimensional assistance

[Hemenway et al., PRA (2013)]
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"linear objective function with semidefinite constraints"

- We give a **converging hierarchy of semidefinite programming (sdp) relaxations**:

$$p_{\text{succ}}(W,k) \leq p^*_{\text{succ}}(W,k) = \text{sdp}_\infty(W,k) \leq \ldots \leq \text{sdp}_1(W,k)$$ <— *efficiently computable!*

Hierarchy of semidefinite programming (sdp) relaxations

First level semidefinite programming relaxation (I)

- Quantum bilinear program:

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$$\text{with } [E(x|i),D(i|y)] = 0$$

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idea: relaxation of this bilinear form

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*motivated by: "**NPA hierarchy**" (Bell inequalities)*

[Lasserre, SIAM (2001)], [Parrilo, Math. Program. (2003)], [Navascues et al., PRL (2007)], [Doherty et al., IEEE CCC (2008)], [Navascues et al., NJP (2008)], [Pironio et al., SIAM (2010)]

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- First step: see as the part of the upper-right block of the Gram matrix

$$\Omega = \sum_{u,v}\langle\psi|X_u X_v|\psi\rangle|u\rangle\langle v| \quad \text{with} \quad X_u = \begin{cases} E(x|i) & u=(i,x) \\ D(j|y) & u=(j,y)\end{cases}$$

for i=j

$$\Omega = \left(\begin{array}{c:c} \langle\psi|E(x|i)\cdot E(x'|i')|\psi\rangle & \langle\psi|E(x|i)\cdot D(j|y)|\psi\rangle \\ \hdashline \langle\psi|E(x'|i')\cdot D(j'|y')|\psi\rangle & \langle\psi|D(j|y)\cdot D(j'|y')]|\psi\rangle \end{array}\right)$$

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- Original constraints can be formulated as positivity conditions on Ω : $\mathrm{sdp}_1(W,k)$

---

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\text{sdp}_1(W,k) = \underset{\Omega}{\text{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\Omega_{(i,x),(i,y)} \\
\text{subject to} \quad & \Omega \in \text{Pos}(1+k|X|+k|Y|)\,, \quad \Omega_{\emptyset,\emptyset} = 1 \quad \text{with } \emptyset \text{ the empty symbol} \\
& \Omega_{u,v} \geq 0 \quad \forall u,v \in X\times[k]\cup Y\times[k]\cup\{\emptyset\} \\
& \sum_x \Omega_{w,(i,x)} = \Omega_{w,\emptyset} \quad \forall i\in[k]\,, w\in X\times[k]\cup Y\times[k]\cup\{\emptyset\} \\
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- Going back to our example:

$$p_{\text{succ}}(Z,2) = \frac{5}{6} \approx 0.833$$

(known before, with two-dimensional assistance)

$$p^*_{\text{succ}}(Z,2) \geq \frac{2+2^{-1/2}}{3} \approx 0.902$$

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\end{aligned}$$

- Going back to our example:

$$p_{\mathrm{succ}}(Z,2) = \frac{5}{6} \approx 0.833$$

(known before, with two-dimensional assistance)

$$p^*_{\mathrm{succ}}(Z,2) \geq \frac{2+2^{-1/2}}{3} \approx 0.902$$

$$Z = \begin{pmatrix} 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix}$$

- Relaxation: $p^*_{\mathrm{succ}}(Z,2) \leq \mathrm{sdp}_1(Z,2) \approx 0.908 = \frac{1}{2} + \frac{1}{\sqrt{6}}$
- Four-dimensional assistance: $p^*_{\mathrm{succ}}(Z,2) \geq \frac{1}{2} + \frac{1}{\sqrt{6}}$

First level semidefinite programming relaxation (II)

- First level relaxation: $p_{\mathrm{succ}}(W,k) \leq p^*_{\mathrm{succ}}(W,k) \leq \mathrm{sdp}_1(W,k)$

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new condition —> $$\Omega_{u,v} \geq 0 \quad \forall u,v \in X\times[k] \cup Y\times[k] \cup \{\emptyset\}$$

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—> further work [Barman and Fawzi., arXiv:1508.04095]

General Quantum Bilinear Optimisation

… and more applications

General Quantum Bilinear Optimisation

$$
\begin{aligned}
p^*_{\mathrm{succ}}(W,k) := \underset{(\mathcal{H},\psi,E,D)}{\mathrm{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\langle\psi|E(x|i)\otimes D(i|y)|\psi\rangle \\
\text{subject to} \quad & \sum_x E(x|i) = 1_{\mathcal{H}} \quad \forall i\in[k]\,, \quad \sum_i D(i|y) = 1_{\mathcal{H}} \quad \forall y \in Y \\
& 0 \le E(x|i) \le 1_{\mathcal{H}}\ \forall (x,i)\in X\times[k]\,, \quad 0\le D(i|y)\le 1_{\mathcal{H}}\ \forall (i,y)\in[k]\times Y\,.
\end{aligned}
$$

… and more applications

General Quantum Bilinear Optimisation

idea: relaxation of this bilinear form

$$p^*_{\text{succ}}(W,k) := \underset{(\mathcal{H},\psi,E,D)}{\text{maximize}} \quad \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\langle\psi|E(x|i)\otimes D(i|y)|\psi\rangle \quad \leq \langle\psi|E(x|i)\cdot D(i|y)|\psi\rangle$$

$$\text{with } [E(x|i), D(i|y)] = 0$$

$$\text{subject to} \quad \sum_x E(x|i) = 1_{\mathcal{H}} \quad \forall i\in[k]\,, \quad \sum_i D(i|y) = 1_{\mathcal{H}} \quad \forall y \in Y$$

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… and more applications

Example:

Randomness

(vs.)

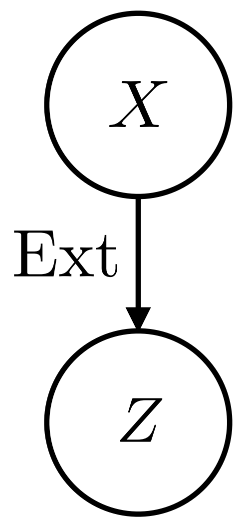
Quantum-Proof Randomness

Quantum Cryptography (I)

- Privacy amplification:

weak source of randomness $X \in \{0,1\}^n$

(with min-entropy k)



uniform random bits $Z \in \{0,1\}^m$

(up to $\epsilon \geq 0$)

- Example: two-universal hashing

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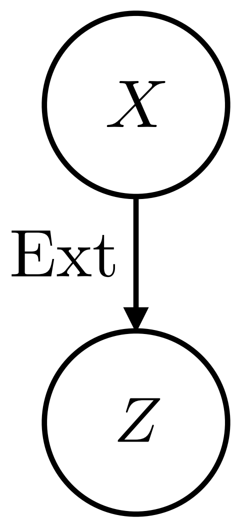

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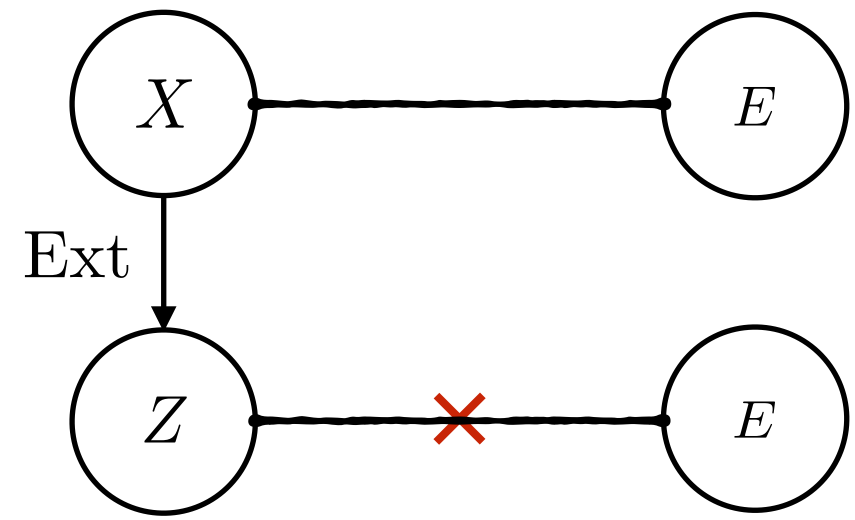
- Example: two-universal hashing

- What happens for quantum adversaries?

weak source of randomness relative to E



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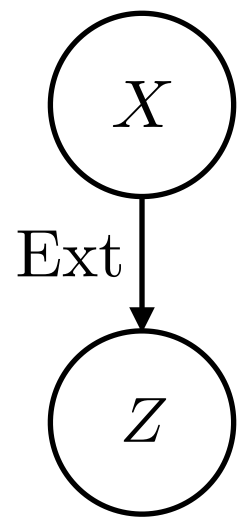

"randomness extractor"

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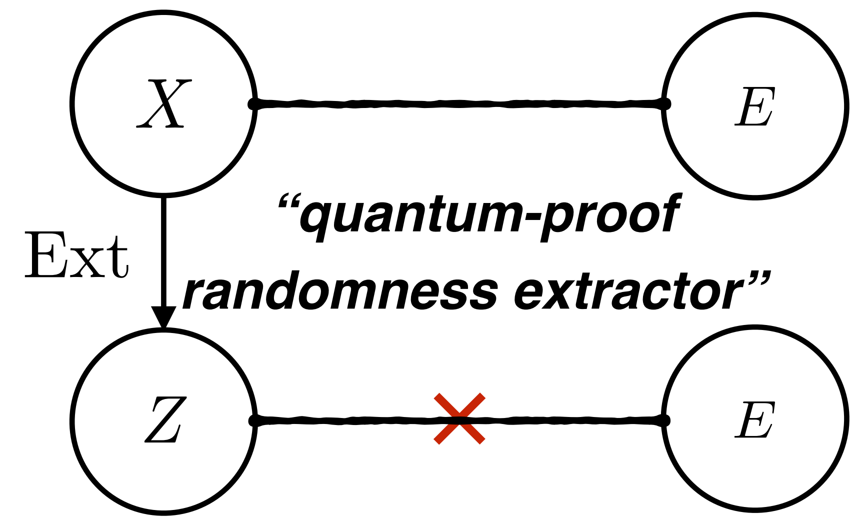

"quantum-proof randomness extractor"

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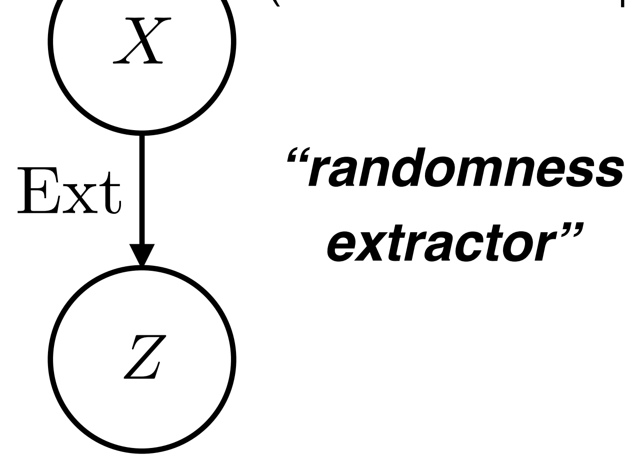

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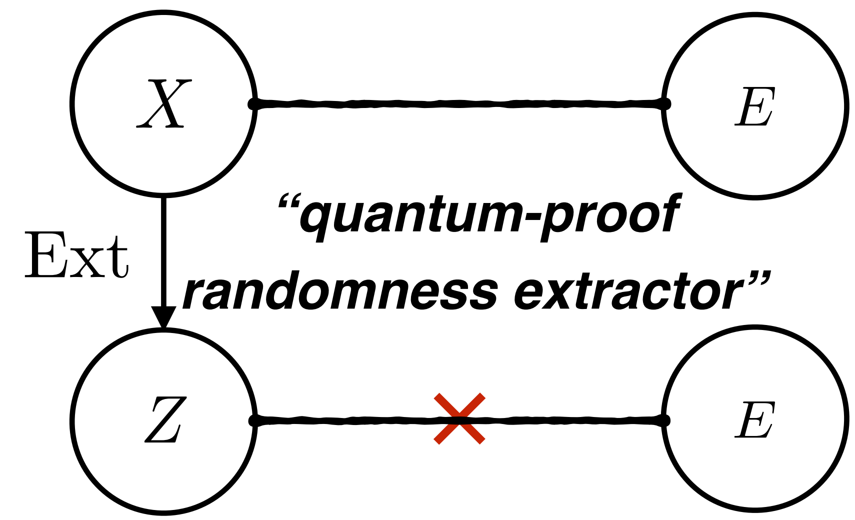

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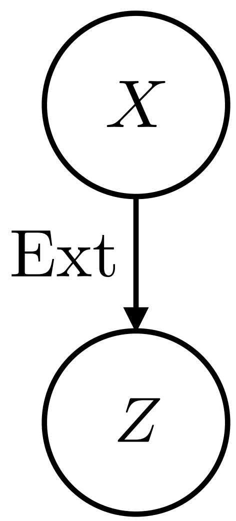

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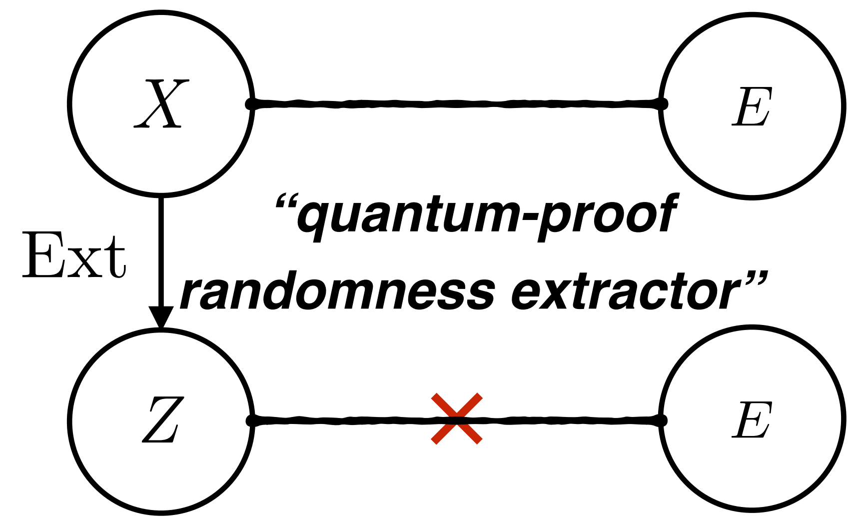

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- Classical versus quantum error (the ϵ): $C(\mathrm{Ext}, k)$ versus $Q(\mathrm{Ext}, k)$ *computable?*

Quantum Cryptography (II)

- For randomness extractors: classical versus quantum adversaries

$C(\mathrm{Ext}, k)$ versus $Q(\mathrm{Ext}, k)$ *computable?*

classical bilinear optimisation	versus	quantum bilinear optimisation
scalar variables	versus	matrix variables

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- For randomness extractors: classical versus quantum adversaries

$C(\mathrm{Ext},k)$ versus $Q(\mathrm{Ext},k)$ *computable?*

classical bilinear optimisation versus quantum bilinear optimisation
scalar variables versus matrix variables

- Converging hierarchy of semidefinite programming relaxations:

$C(\mathrm{Ext},k) \leq Q(\mathrm{Ext},k) = \mathrm{sdp}_{\infty}(\mathrm{Ext},k) \leq \ldots \leq \mathrm{sdp}_1(\mathrm{Ext},k)$ <— *efficiently computable!*

—> upper bounding the power of quantum adversaries

see arXiv:1506.08810 for more (including references)

Conclusion / Outlook

- Understand **quantum assistance** (noisy channel coding) and **quantum adversaries** (randomness extractors) using optimisation methods

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- First outer hierarchy for optimisations over the **completely positive semidefinite cone** (symmetric matrices that admit a Gram representation by positive semidefinite matrices, [Laurent and Piovesan, arXiv:1312.6643]) —> quantum graph parameters

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- Any other ideas what to quantise?

Thanks!