On the quantum capacity of the qubit depolarizing channel

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Qubit depolarizing channel

- Assume that we have a qubit in a state ρ.
- Fixing some basis $\{|0\rangle, |1\rangle\}$, we consider the following errors:

error	action	operator
bit flip	$\|0\rangle \mapsto \|1\rangle$ $\|1\rangle \mapsto \|0\rangle$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
phase flip	$\|0\rangle \mapsto \|0\rangle$ $\|1\rangle \mapsto -\|1\rangle$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
combined flip	$\|0\rangle \mapsto i\|1\rangle$ $\|1\rangle \mapsto -i\|0\rangle$	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Qubit depolarizing channel

- **Depolarizing channel** $\mathcal{D}_p$: each of the three errors (bit flip, phase flip, combined flip) occurs with the same probability $p/3$:

$$\mathcal{D}_p(\rho) := (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).$$

- Alternatively: $\mathcal{D}_p$ replaces the input with the completely mixed state $\pi_2 := \frac{1}{2} I_2$ with ”probability” $q = 4p/3$:

$$\mathcal{D}_p(\rho) = (1-q)\rho + q\pi_2.$$

- Exact expression for classical capacity of $\mathcal{D}_p$ is known. [King 2003]

- Exact expression for **quantum capacity of $\mathcal{D}_p$ is unknown**.

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Entanglement generation

- Alice and Bob are connected by a quantum channel $\mathcal{N}: A \to B$ that they can use n times.
- **Goal:** Generate entanglement in the form of m_n ebits $|\Phi\rangle \sim |00\rangle + |11\rangle$.
- **Protocol:** Alice prepares a pure state in her lab, sends one half through n copies of $\mathcal{N}$, Bob applies some decoding.
- **Quantum capacity** $Q(\mathcal{N})$: largest possible rate at which ebits can be generated with vanishing error.

Quantum capacity

- **LSD formula** [Lloyd 1997; Shor 2002; Devetak 2005]:

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}), \quad (*)$$

  where **channel coherent information** $Q^{(1)}(\mathcal{N})$ is defined as

$$Q^{(1)}(\mathcal{N}) := \max_{|\psi\rangle_{A'A}} I(A'\rangle B)_{(\mathrm{id} \otimes \mathcal{N})(\psi)},$$

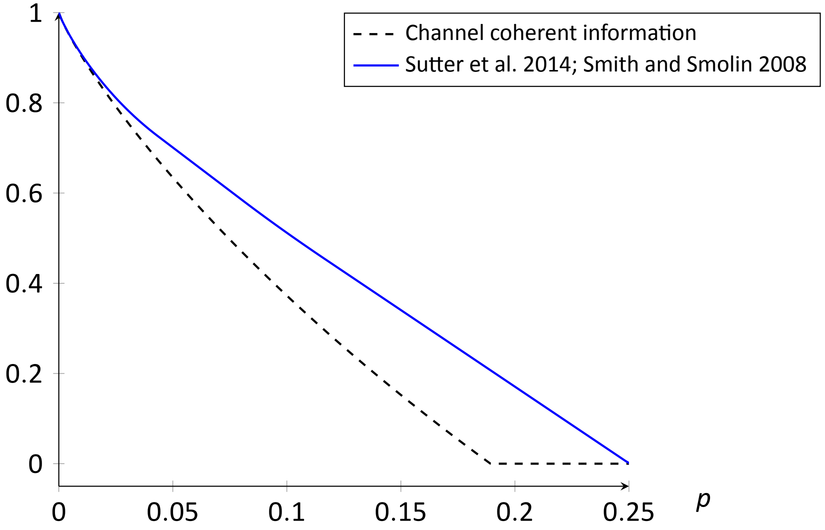
  with the coherent information $I(A\rangle B)_\rho = S(B)_\rho - S(AB)_\rho$.

- **Hashing bound:** $Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N})$ [Devetak and Winter 2005]

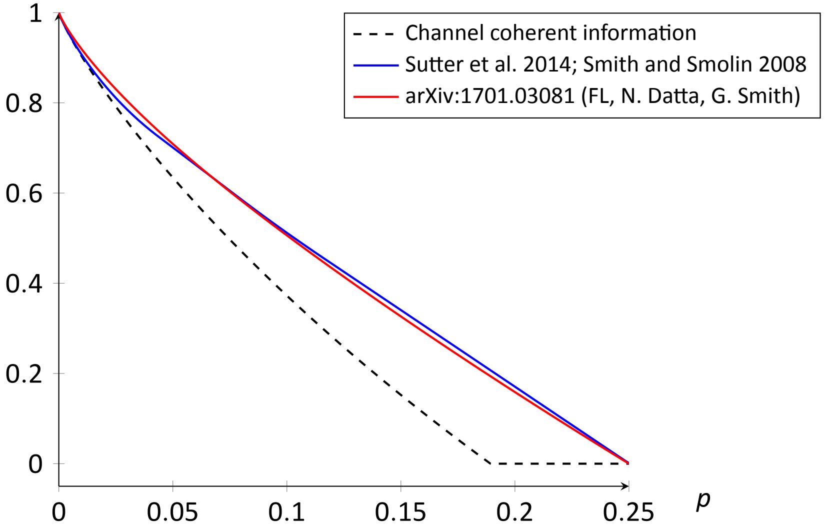
- **Regularized formula** $(*)$ is in general **intractable to compute**, in particular for $\mathcal{D}_p$.

- Hence, we are interested in lower and upper bounds on $Q(\mathcal{D}_p)$.

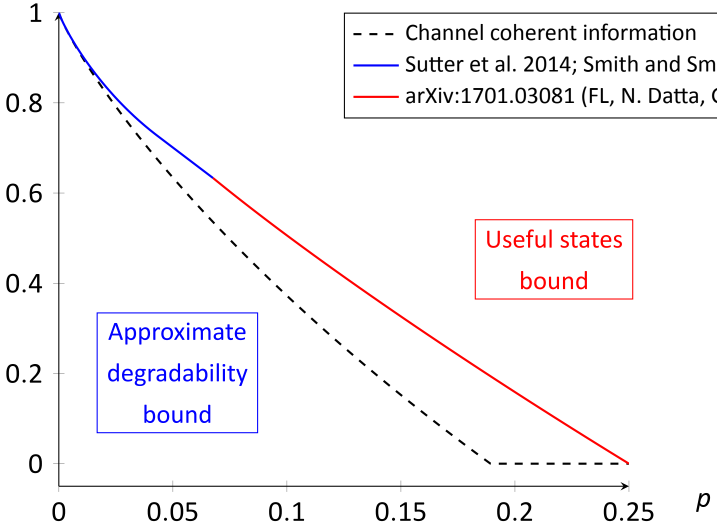
Upper and lower bounds on $Q(\mathcal{D}_p)$



Upper and lower bounds on $Q(\mathcal{D}_p)$



Upper and lower bounds on $Q(\mathcal{D}_p)$



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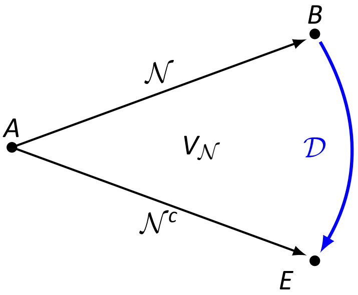
Low depolarizing noise

- For low depolarizing noise, i.e., p close to 0, the depolarizing channel is ”almost” the identity channel:

$$\mathcal{D}_p = (1-p)\,\mathrm{id} + \tfrac{p}{3}(\text{Pauli errors})$$

- Capacity of the identity channel is known: $Q(\mathrm{id}) = \log d$.
- Particularly simple example of a **degradable channel**:



degradable:

$\exists \mathcal{D}: B \to E$ s.t.

$\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$

$V_{\mathcal{N}} : A \to BE$ is an isometry s.t.

$$\mathcal{N} = \mathrm{Tr}_E(V \cdot V^\dagger),$$
$$\mathcal{N}^c = \mathrm{Tr}_B(V \cdot V^\dagger).$$

Degradability and quantum capacity

- Recall: Channel coherent information

$$Q^{(1)}(\mathcal{N}) := \max_{|\psi\rangle_{A'A}} I(A'\rangle B)_{(\mathrm{id}\otimes\mathcal{N})(\psi)}.$$

- Recall: LSD-formula

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}).$$

- **Degradable channels:** $Q^{(1)}(\cdot)$ is weakly additive,

$$Q^{(1)}(\mathcal{N}^{\otimes n}) = nQ^{(1)}(\mathcal{N}) \quad \Longrightarrow \quad Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \qquad (*)$$

[Devetak and Shor 2005]

- **Idea:** Since $\mathcal{D}_p$ is degradable for $p = 0$, is $(*)$ approximately true for $p \gtrsim 0$?

Continuity of quantum capacity

- What does it mean for two channels to be close?
- **Diamond norm:** For a linear map Φ with d-dimensional input,

$$\|\Phi\|_\diamond := \sup_X \frac{\|(\mathrm{id}_d \otimes \Phi)(X)\|_1}{\|X\|_1}.$$

- Operational meaning: $\frac{1}{2}\|\Phi - \Psi\|_\diamond$ measures distinguishability of channels Φ and Ψ (similar to trace distance and states).
- Quantum capacity is continuous with respect to the diamond norm [Leung and Smith 2009]:

If $\|\mathcal{N} - \mathcal{M}\|_\diamond \leq \epsilon$, then $|Q(\mathcal{N}) - Q(\mathcal{M})| \leq 8\epsilon \log |B| + 4h(\epsilon)$,

where $h(\epsilon)$ is the binary entropy.

Low depolarizing noise and approximate degradability

- Since $\|\mathcal{D}_p - \mathrm{id}\|_\diamond \leq p$, we expect $Q(\mathcal{D}_p) \lesssim 1$ for small p.
- However, continuity bound is not useful for quantitative bounds.
- Recall: $\mathcal{N}$ is degradable if there is a $\mathcal{D}$ such that $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$.
- **Approximate degradability** [Sutter et al. 2014]

  A channel $\mathcal{N}$ is η-degradable if there is another channel $\mathcal{D}$ s.t.

  $$\|\mathcal{N}^c - \mathcal{D} \circ \mathcal{N}\|_\diamond \leq \eta.$$

- **Degradability parameter** $\mathrm{dg}(\mathcal{N}) :=$ optimal such η
- $\mathrm{dg}(\mathcal{N})$ is the solution of an SDP (also yields optimal $\mathcal{D}$).

Approximate degradability and quantum capacity

- **Hashing bound:** $Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N})$ [Devetak and Winter 2005]
- This is optimal for **degradable channels**: [Devetak and Shor 2005]

$$Q^{(1)}(\mathcal{N}^{\otimes n}) = nQ^{(1)}(\mathcal{N}) \quad \Longrightarrow \quad Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$$

- ***η*-degradable channels:**

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + f(\eta, |E|),$$

where $f(\eta, |E|) = -\eta \log \eta + O(\eta)$.

- Recovers degradable case: $f(\eta, |E|) \xrightarrow{\eta \to 0} 0$.
- **Problem:** In the generic case, this is not a very useful approximation.

Approximate degradability and quantum capacity

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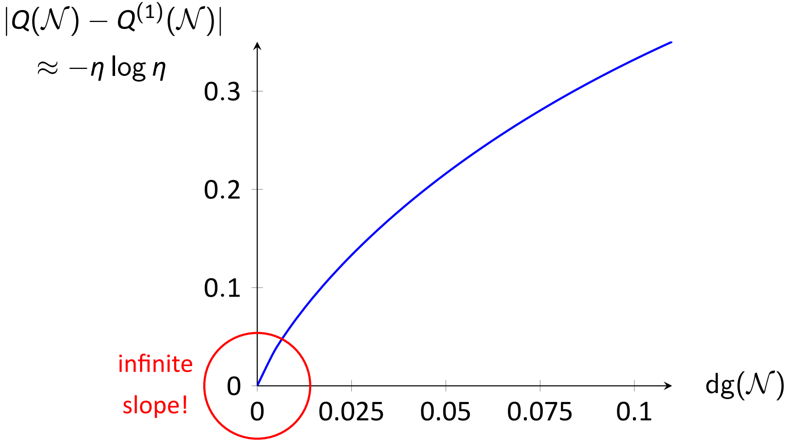
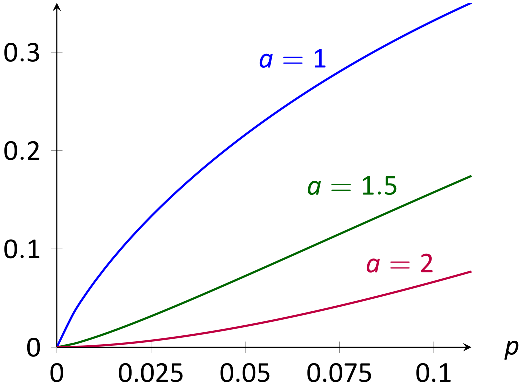


Approximate degradability and quantum capacity

- **But:** If $\mathrm{dg}(\mathcal{N}) = p^a$ for some underlying parameter p, and $a > 1$
  $\longrightarrow$ approximation is tangent!

$|Q(\mathcal{N}) - Q^{(1)}(\mathcal{N})|$
$\approx -p^a \log p^a$



Main results

- **Degradability:** $\mathrm{dg}(\mathcal{D}_p) = O(p^2)$.
- **Quantum capacity:**

$$Q(\mathcal{D}_p) = 1 - h(p) - p\log 3 + O(p^2 \log p^2).$$

- **Proof idea:** Guess degrading map $\mathcal{M}_p$ in $\|\mathcal{D}_p^c - \mathcal{M}_p \circ \mathcal{D}_p\|_\diamond$.
- **Intuition:** For small p, depolarizing channel is almost identity, so complementary channel is almost completely depolarizing.
- Let's use the **complementary channel** itself **as degrading map** $\mathcal{M}_p$, and "give a little more to Bob":

$$\mathcal{M}_p = \mathcal{D}_{s(p)}^c \quad \text{with } s(p) = p + ap^2.$$

- For $a = 8/3$, this yields $\|\mathcal{D}_p^c - \mathcal{D}_{s(p)}^c \circ \mathcal{D}_p\|_\diamond = O(p^2)$.

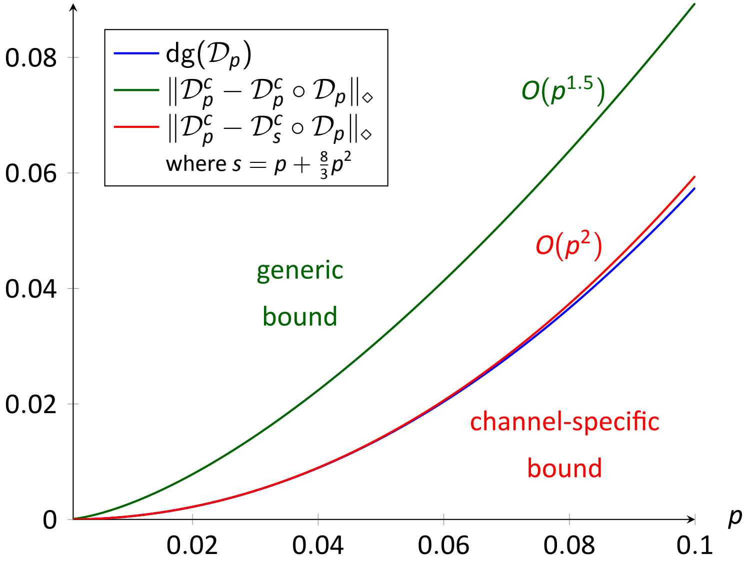
Usefulness of approximate degradability bound

- Approximate degradability might be useless in generic case.
- **However:** Extremely useful for channels with some underlying noise parameter such as $\mathcal{D}_p$, for which $\mathrm{dg}(\mathcal{D}_p) = O(p^2)$.
- **Generalizations:**
  - Same quadratic ansatz works for any generalized Pauli channel $\mathcal{N}$, proving $\mathrm{dg}(\mathcal{N}) = O(p^2)$.
  - For any "low-noise channel" $\mathcal{N}$ with $\|\mathrm{id} - \mathcal{N}\|_\diamond \leq \epsilon$, we have

    $$\|\mathcal{N}^c - \mathcal{N}^c \circ \mathcal{N}\|_\diamond \leq 2\epsilon^{1.5}.$$

    $\longrightarrow$ Low-noise channels are approximately degraded by their complementary channel!

Approximate degradability of depolarizing channel



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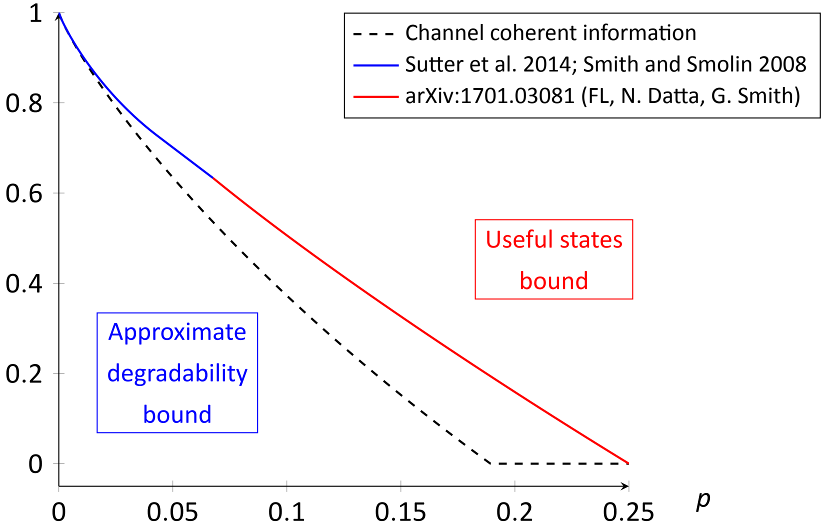
Upper and lower bounds on $Q(\mathcal{D}_p)$



Detour: Entanglement distillation using forward LOCC

- **Entanglement distillation:** Convert ”noisy” entanglement resource ρ_{AB} shared between Alice and Bob into a ”clean” form, e.g., ebits $|\Phi\rangle \sim |00\rangle + |11\rangle$.
- **Protocol**: local operations (LO) and classical communication (CC) between Alice and Bob.
- Recall that forward CC **does not increase** quantum capacity of a channel [Bennett et al. 1996; Barnum et al. 2000].
- Restrict to **forward CC** from Alice to Bob **only**.
- **Distillable entanglement** $D_{\rightarrow}(\rho_{AB})$: largest possible rate at which ebits can be distilled with vanishing error.

Entanglement distillation and entanglement generation

- **Entanglement distillation** and **entanglement generation** are just the **static** and **dynamic** side of the same coin! [Bennett et al. 1996]

- **egen** $\rightarrow$ **edist**:
  - Shared state ρ_{AB} + forward LOCC $\longrightarrow$ teleportation scheme [Bennett et al. 1993] $\longrightarrow$ noisy channel $\mathcal{N}_\rho$.
  - egen code for $\mathcal{N}_\rho$ distills ebits from ρ_{AB} $\longrightarrow$ edist protocol.

- **edist** $\rightarrow$ **egen**:
  - ebit (in Alice's lab) + channel $\mathcal{N}$ from Alice to Bob $\rightarrow$ shared Choi state $\tau_{\mathcal{N}}$.
  - edist protocol for $\tau_{\mathcal{N}}$ generates ebits through $\mathcal{N}$ $\longrightarrow$ egen code.

Entanglement distillation and entanglement generation

- These two mappings (egen $\leftrightarrow$ edist) are inverse to each other for so-called **teleportation-simulable** (TS) channels.

- **TS channel:** any output state can be obtained using a teleportation protocol run on the Choi state of the channel.

- Depolarizing channel is TS [Bennett et al. 1996]:

$$Q(\mathcal{D}_p) = D_{\rightarrow}(\tau_{\mathcal{D}_p})$$

- From now on, focus on the ”static” resource $\tau_{\mathcal{D}_p}$ and $D_{\rightarrow}(\tau_{\mathcal{D}_p})$.

Useful states for entanglement distillation

- **Hashing bound** [Devetak and Winter 2005]

$$D_{\rightarrow}(\rho_{AB}) \geq I(A\rangle B)_{\rho}.$$

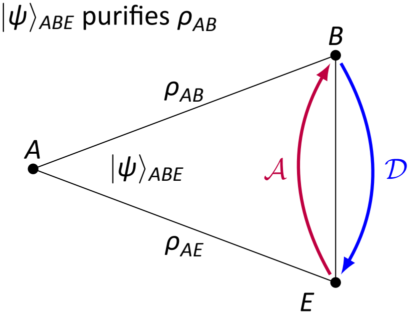
- **But:** $D_{\rightarrow}(\cdot)$ is also given by a regularized formula [Devetak and Winter 2005], and hence as **intractable to compute** as $Q(\cdot)$.
- **Solution:** Identify "useful" and "useless" states which make the quantity $D_{\rightarrow}(\cdot)$ behave nicely.
- **Useful states:** States ρ_{AB} for which hashing is optimal protocol:

$$D_{\rightarrow}(\rho_{AB}) = I(A\rangle B)_{\rho} \geq 0.$$

- **Useless states:** Undistillable states σ_{AB}, i.e., $D_{\rightarrow}(\sigma_{AB}) = 0$. They always have $I(A\rangle B)_{\sigma} \leq 0$.

Useful states for entanglement distillation

- The **useful** and **useless** states for one-way entanglement distillation are given by **degradable** and **antidegradable** states.



degradable:

$\exists \mathcal{D}: B \to E$ s.t.

$$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$$

antidegradable:

$\exists \mathcal{A}: E \to B$ s.t.

$$\rho_{AB} = (\mathrm{id}_A \otimes \mathcal{A})(\rho_{AE})$$

Useful states for entanglement distillation

- **Key observation**: $D_{\rightarrow}(\cdot)$ is convex on degradable and antidegradable states!
  (proved by [Wolf and Pérez-García 2007] for quantum capacity)
- **Main result:** Decompose ρ_{AB} into mixture of degradable (ω_i) and antidegradable (σ_i) states,

$$\rho_{AB} = \sum_i p_i \omega_i + \sum_i p_i \sigma_i, \qquad (*)$$

  then we have the following upper bound:

$$D_{\rightarrow}(\rho_{AB}) \leq \sum_i p_i \, I(A\rangle B)_{\omega_i}. \qquad (**)$$

- $(*)$ always exists, since **every pure state is degradable**.
- Minimizing $(**)$ over all pure-state decompositions of ρ_{AB} yields **entanglement of formation** $E_F(\rho_{AB})$.

Quantum capacity of depolarizing channel

- **Goal:** Apply this upper bound to Choi state $\tau_{\mathcal{D}_p}$ of depolarizing channel to get upper bound on $Q(\mathcal{D}_p) = D_{\rightarrow}(\tau_{\mathcal{D}_p})$.
- **Strategy:** Use symmetries of $\tau_{\mathcal{D}_p}$!
- Choi state $\tau_{\mathcal{D}_p}$ of depolarizing channel is an **isotropic state**: 1-parameter family of states $\{\mathrm{Iso}(f) : f \in [0,1]\}$ invariant under all local unitaries of the form $U \otimes U^*$. (here, $f = 1 - p$)
- The symmetry group $\{U \otimes U^* : U \text{ unitary}\}$ defines a bilateral twirl operation

$$\rho_{AB} \longmapsto \int dU \, (U \otimes U^*)\rho_{AB}(U \otimes U^*)^\dagger = \mathrm{Iso}(f_\rho)$$

where $f_\rho = \langle\Phi|\rho_{AB}|\Phi\rangle$.

Quantum capacity of depolarizing channel

- By the no-cloning theorem, $Q(\mathcal{D}_p) = 0$ for $p \geq 1/4$.
- For $p \leq 1/4$, the $U \otimes U^*$-symmetry of $\tau_{\mathcal{D}_p}$ allows us to restrict to degradable states that "twirl" to $\tau_{\mathcal{D}_p}$:

  $$D_{\rightarrow}(\tau_{\mathcal{D}_p}) \leq \inf\{I(A\rangle B)_\rho : \rho_{AB} \text{ degradable}, \langle\Phi|\rho_{AB}|\Phi\rangle = 1-p\}.$$

- **Bad news:** This is a non-convex optimization problem, since the set of degradable states is not convex.
- **Good news:** We can solve it numerically for low dimensions (qubits, qutrits) $\longrightarrow$ yields advertised upper bound for $d = 2$!

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Conclusion

- Quantum capacity of depolarizing channel is unknown.
- Best upper bound for low depolarizing noise: Approximate degradability bound.
- Approximates desirable properties of degradable channels.
- Best upper bound for high depolarizing noise: Useful states bound.
- Based on decomposition of Choi state into degradable and antidegradable parts.

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Thank you very much for your attention!